On Eccentric Domination in Graphs
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Abstract: A set $D \subseteq V(G)$ is an eccentric dominating set if $D$ is a dominating set of $G$ and for every $v \in V-D$, there exists at least one eccentric vertex of $v$ in $D$. The minimum cardinality of eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. In this paper, we have provided some new bounds for $\gamma_{ed}(G)$ and established the relation between $\gamma_{ed}(G)$, $\alpha_0(G)$ and $\beta_0(G)$. We have also characterized graphs for which $\gamma_{ed}(G) = p-1$ and $p-2$.

Keywords: eccentric dominating set, minimum eccentric dominating set.

1. Introduction

Graphs discussed in this paper are undirected and simple. For graph theoretic terminology refer to Harary[5], Buckley and Harary[3]. For a graph, let $V(G)$ and $E(G)$ denotes its vertex and edge set respectively. A graph with $p$ vertices and $q$ edges is called a $(p, q)$ graph.

The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The length of any shortest path between any two vertices $u$ and $v$ of a connected graph $G$ is called the distance between $u$ and $v$ and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be $\infty$. For a connected graph $G$, The eccentricity $e(v)$ of $v$ is the distance to a vertex farthest from $v$. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The radius $\text{rad}(G)$ is the minimum eccentricity of the vertices, whereas the diameter $\text{diam}(G)$ is the maximum eccentricity.

A vertex cover of a graph $G$ is a set of vertices that covers all the edges. The vertex covering number $\alpha_0(G)$ of $G$ is minimum cardinality of a vertex cover.

A set $S$ of vertices of $G$ is independent if no two vertices in $S$ are adjacent. The independent number $\beta_0(G)$ of $G$ is the maximum cardinality of an independent set.

The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters.
A set $D \subseteq V$ is said to be a dominating set in $G$, if every vertex in $V - D$ is adjacent to some vertex in $D$. The cardinality of minimum dominating set is called the domination number and is denoted by $\gamma(G)$. For details on $\gamma(G)$, refer to [4].

Janakiraman, Bhanumathi and Muthammai [2, 6] introduced and studied the concept of eccentric dominating set. In [1], they have studied the eccentric domination in trees. A set $D \subseteq V(G)$ is an eccentric dominating set if $D$ is a dominating set of $G$ and for every $v \in V - D$, there exists at least one eccentric vertex of $v$ in $D$. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. An eccentric dominating set with cardinality $\gamma_{ed}(G)$ is known as minimum eccentric dominating set. An eccentric dominating set $D$ is a minimal eccentric dominating set if no proper subset $D^* \subseteq D$ is an eccentric dominating set.

The following results are needed to study the eccentric dominating set of a graph $G$.

**Theorem: 1.1** [5]: For any graph $G$ with even order $n$ and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of $G$ are the cycle $C_4$ or the corona $H \circ K_1$ for any connected graph $H$.

**Theorem: 1.2** [5]: If $G$ is a connected graph with order $n$ and $\delta(G) \geq 2$ and $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$, then $G \in A \cup B$.

**Theorem: 1.3** [5]: $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

**Theorem: 1.4** [5]:

$$
\gamma(C_n) = \begin{cases} 
\left\lceil \frac{n}{3} \right\rceil & \text{if } n = 3k + 3, \\
\left\lceil \frac{n}{3} \right\rceil & \text{if } n = 3k + 1 \text{ or } 3k + 2.
\end{cases}
$$
Theorem: 1.5\cite{7}: $\gamma_{ed}(K_n) = 1$.

Theorem: 1.6\cite{7}: $\gamma_{ed}(K_m, n) = 2$.

Theorem: 1.7\cite{7}: $\gamma_{ed}(P_n) = \gamma(P_n)$ or $\gamma(P_n) + 1$.

Theorem: 1.8\cite{7}: (i) $\gamma_{ed}(C_n) = \frac{n}{2}$ if $n$ is even.

(ii) $\gamma_{ed}(C_n) = 3\left(\frac{n}{3}\right)$ if $n = 3m + 1$ and is odd.

(iii) $\gamma_{ed}(C_n) = \frac{n}{3} + 1$ if $n = 3m + 2$ and is odd.

Theorem: 1.9\cite{3}: If $G$ is a connected graph with $n$ vertices then $\gamma_{ed}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.

Theorem: 1.10\cite{7}: $\gamma_{ed}(C_4) = 2$, $\gamma_{ed}(C_5) = 3$ and $\gamma_{ed}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$, $n \geq 6$.

2. Some New Results on Eccentric Domination in Graphs

Let $G$ be a $(p, q)$ graph. First, we shall find the relation between $\gamma_{ed}(G)$, $\alpha_0(G)$ and $\beta_0(G)$.

Lemma: 2.1: Let $G$ be a connected graph with $\text{rad}(G) = 1$ and $\text{diam}(G) = 2$. Then $\gamma_{ed}(G) \leq \frac{p - \beta_0 + 3}{2} = \frac{\alpha_0 + 3}{2}$.

Proof: If $G$ is connected, then $\gamma(G) \leq \beta_0(G)$. Any maximum independent set dominate the graph $G$. Let $S$ be a maximum independent set. Then $S$ is a dominating set and $V - S$ is also a dominating set.

Let $G$ be a connected graph with $\text{rad}(G) = 1$ and $\text{diam}(G) = 2$. If $u \in S$, then it is adjacent to at least one element of $S$. Suppose $v$ is adjacent to every element of $S$ and if $e(v) = 1$, then $u \in S$ is an eccentric vertex of $v$. If $e(v) = 2$ and is adjacent to every element of $S$, then eccentric vertex of $v$ is in $V - S$. Hence, at most $\left\lfloor \frac{|V - S|}{2} \right\rfloor$ vertices from $V - S$ are needed to dominate $G$ eccentrically. Thus, $\gamma_{ed}(G) \leq \beta_0(S) + \frac{p - \beta_0}{2} = \frac{p + \beta_0}{2}$.

$S$ is a dominating set. Let $v \in V - S$ such that $e(v) = 1$. $v$ dominates $G$. Since $S$ is independent, any $w \in S$ is eccentric to all the vertices of $S$. Consider the remaining
Let D be a subset of V−S, which contains eccentric vertices of elements of V−S. Then \( \{u, w\} \cup D \) is an eccentric dominating set of G. Thus, \( \gamma_{ed}(G) \leq \frac{p-1-\beta_0}{2} + 1 \). That is \( \gamma_{ed}(G) \leq \frac{p-\beta_0+3}{2} \).

V−S is also a dominating set for G and w \( \in S \) is eccentric to all vertices of S. Hence, \( (V−S) \cup \{w\} \) is an eccentric dominating set of G. Thus, \( \gamma_{ed}(G) \leq p-\beta_0+1 \).

So, \( \gamma_{ed}(G) \leq \min\left\{ \frac{p+\beta_0}{2}, p-\beta_0 + 1, \frac{p-\beta_0+3}{2} \right\} \).

Hence, \( \gamma_{ed}(G) \leq \frac{p-\beta_0+3}{2} = \frac{\alpha_0+3}{2} \).

This bound is sharp, for the following graph G.

**Example: 2.1:**

\[ \gamma_{ed}(G) = 3, \beta_0(G) = 3, p = 6. \]

**Lemma: 2.2:** Let G be a 2-self-centered graph. Then \( \gamma_{ed}(G) \leq \alpha_0(G) \).

**Proof:** Let S be a maximum independent set. D = V−S dominates all the vertices of G and u \( \in S \) is eccentric to all other vertices of S. Hence, \( \gamma_{ed}(G) \leq |D|+1 = p-\beta_0+1 = \alpha_0+1 \).

**Case (i):** If V−S is also independent, then G is a bipartite graph. Also, G is 2-self-centered. Hence, G is a complete bipartite graph. Therefore, \( \gamma_{ed}(G) = 2 \).

**Case (ii):** If V−S is not independent, there exists u, v \( \in V−S \) such that uv \( \in E(G) \) and every vertex in S is adjacent to at least two vertices of V−S. Hence, \( \gamma(G) \leq |D|-1 \). This implies that \( \gamma_{ed}(G) \leq |D|-1+1 = |D| \). Therefore, \( \gamma_{ed}(G) \leq p-\beta_0 = \alpha_0 \). Hence, in all the cases, \( \gamma_{ed}(G) \leq \alpha_0 \) where G is 2 self-centered.
Example: 2.2:
Let $G = C_5$. Let $v_1, v_2, v_3, v_4, v_5$ represent the cycle $C_5$. $D_1 = \{v_1, v_3, v_4\}$ is a minimum eccentric dominating set and also a vertex covering of $G$. $\gamma_{ed}(G) = \alpha_o(G) = 3$.

Lemma 2.3: Let $G$ be a graph with diam($G$) > 2. Then $\gamma_{ed}(G) \leq \min \left\{ \frac{p + \alpha_o}{2}, \frac{p + \beta_o}{2} \right\}$.

Proof: Let $G$ be a graph with diam($G$) > 2. Let $S$ be a maximum independent set, $S$ is a dominating set and $V-S$ is also a dominating set. Thus, $\gamma_{ed}(G) \leq \beta_o + \frac{p - \beta_o}{2}$ and $\gamma_{ed}(G) \leq (p + \alpha_o)/2$.

Therefore, $\gamma_{ed}(G) \leq \frac{p + \beta_o}{2}$ and $\gamma_{ed}(G) \leq \frac{p + \alpha_o}{2}$.

Hence, $\gamma_{ed}(G) \leq \min \left\{ \frac{p + \alpha_o}{2}, \frac{p + \beta_o}{2} \right\}$.

Example: 2.3:
$$D_1 = \{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{17}, v_{18}\}$$ is a minimum eccentric dominating set and also an independent set of $G$. $\gamma_{ed}(G) = \beta_o(G) = 12 = (p + \alpha_o)/2$.

$D_2 = \{v_1, v_4, v_7, v_{10}, v_{13}, v_{16}\}$ is a vertex covering of $G$. $\alpha_o(G) = 6$.

From Lemmas 2.1, 2.2 and 2.3, we have the following theorem:

**Theorem 2.1:** For any connected graph $G$, $\gamma_{ed}(G) \leq \min \left\{ \frac{p + \alpha_o}{2}, \frac{p + \beta_o}{2} \right\}$.

Bounds of $\gamma_{ed}(G)$ in terms of number of vertices of $G$ is given in the following lemmas.

**Lemma 2.4:** Let $G$ be a graph with radius one and diameter two. Then, $\gamma_{ed}(G) \leq p/2$.

**Proof:** Let $G$ be a graph with radius one and diameter two. Consider the following cases:
Case (i): $G$ has a pendent vertex

Then clearly $\gamma_{ed}(G) = 2$. Thus if $p \geq 4$, then $\gamma_{ed}(G) \leq p/2$.

Case (ii): $G$ has no pendent vertex

Let $u$ be a vertex of minimum degree. Then $\text{deg}(u) = \delta(G) = p-2$ and $e(u) = 2$. In this case $\gamma_{ed}(G) \leq ((\delta(G)-1)/2)+1 \leq ((p-2-1)/2)+1 \leq (p-1)/2$, where $t$ is the number of vertices with eccentricity 1.

**Lemma 2.5:** Let $G$ be a self-centered graph of diameter 2 with $p \geq 6$. Then, $\gamma_{ed}(G) \leq p/2$.

**Proof:** Since $G$ is a self-centered graph with diameter 2, degree of any vertex of $G$ is less than or equal to $p-2$. If degree of every vertex of $G$ is equal to $p-2$, then $G = K_{p-1}$ factor and $\gamma_{ed}(G) = p/2$. Hence assume that $\delta(G) \leq p-3$.

Case (i): there exists $u \in V(G)$ such that $\delta(G) = \text{deg}(u) = p-3$.

Since, $\text{deg}(u) = p-3$ there are exactly two vertices $w_1, w_2$ in $N_2(u)$. Also, degree of those vertices is $\geq p-3$.

**Sub case (i):** $w_1, w_2$ in $N_2(u)$ are adjacent.

$\text{deg}(w_1) = p-3$ or $p-2$. Therefore, $w_1$ is adjacent to at least $p-4$ vertices of $N_1(u)$. Similarly, $w_2$ is adjacent to at least $p-4$ vertices of $N_1(u)$. Hence, $w_1$ and $w_2$ are non-adjacent to at most one vertex of $N_1(u)$. Thus $u$ with $(p-3)/2$ vertices of $N_1(u)$ form an eccentric dominating set. Thus, $\gamma_{ed}(G) \leq 1+(p-3)/2 = (p-1)/2$.

**Sub Case (ii):** $w_1, w_2$ in $N_2(u)$ are non adjacent.

In this case, $w_1$ and $w_2$ are adjacent to all the $p-3$ vertices of $N_1(u)$. Hence, again $\gamma_{ed}(G) \leq 1+(p-3)/2 = (p-1)/2$.

Case (ii): there exists $u \in V(G)$ such that $\delta(G) = \text{deg}(u) < p-3$.

Let $\text{deg}(u) = k < p-3$.

**Sub Case (i):** Suppose $k = 2$.

Since $G$ is 2 self-centered, any two vertices of $G$ lie on a common cycle of maximum length four or five. Hence, $w \in N_1(u)$ is adjacent to at least one vertex in $N_1(u)$. Thus, exactly three vertices $\{u\} \cup N_1(u)$ dominate $G$ eccentrically. Hence, $\gamma_{ed}(G) \leq 3 \leq p/2$ when $p \geq 6$.

**Sub case (ii):** Suppose $k > 2$.

Again, since $G$ is 2 self-centered, any two vertices of $G$ lie on a common cycle of maximum length four or five. Also a vertex and its eccentric vertex lie on a common cycle of length 4 or 5. Let $x$ and $y$ be any two adjacent vertices of $u$. These three vertices lie on a common cycle of length four or five. If $u$ and $x$ (or $y$) lie on a cycle of length four, then $\{u, x\}$ dominates the vertices of this cycle eccentrically, otherwise $\{u, x, y\}$ dominates the
vertices of this cycle eccentrically. Hence, if there are more cycles, additionally at most one (in the case of $C_4$) or two (in the case $C_5$) vertices are needed to dominate the vertices of this cycle. So, in general a $\gamma_{ed}$-set of $G$ contains at most $p/2$ vertices. Therefore, $\gamma_{ed}(G) \leq p/2$.

In a similar way, we can prove that $\gamma_{ed}(G) \leq p/2$, if $G$ is any $k$ self-centered graph. Thus, we have the following theorem:

**Theorem: 2.2:** Let $G$ be a self-centered graph with $p \geq 6$. Then, $\gamma_{ed}(G) \leq p/2$.

**Theorem: 2.3:** Let $G$ be a cubic graph on 6 vertices. Then $\gamma(G) = \gamma_{ed}(G) = 2$.

**Proof:** Let $u \in V(G)$. $N_1(u)$ contains three vertices and $N_2(u)$ contains 2 vertices since $\text{deg } u = 3$ for all $u \in V(G)$. Also, $<N_1(u)>$ is not complete, since if it is complete, $v \in N_1(u)$ has no adjacent vertices in $N_2(u)$ for all $v \in N_1(u)$, which is a contradiction.

Now, let $N_2(u) = \{w_1, w_2\}$.

**Case (i):** $<N_2(u)>$ is independent. $w_1$ is not adjacent to $w_2$. Therefore $w_1$ is adjacent to all vertices of $N_1(u)$ and is eccentric to $u$ and $w_2$. Similarly, $w_2$ is adjacent to all of $N_1(u)$ and is eccentric to $u$ and $w_1$. Hence, elements in $N_1(u)$ has eccentric vertices in $N_1(u)$ only and for $v \in N_1(u)$, $v$ is adjacent to $w_1$ and $w_2$ and $v$ is adjacent to $u$. Hence, $\text{deg } v \in N_1(u) = 3-2-1 = 0$. But $N_1(u)$ contains 3 elements. Therefore, $v$ is not adjacent to exactly two elements of $N_1(u)$. Hence, exactly one vertex $w \in N_1(u)$ is an eccentric vertex of elements of $N_1(u)$. Hence, $\{u, w\}$ is an eccentric dominating set of $G$. Therefore, $\gamma_{ed}(G) = 2$.

**Case (ii):** $<N_2(u)> = K_2$. $w_1$ is adjacent to $w_2$. $w_1$ is adjacent to exactly 2 vertices of $N_1(u)$ and $w_2$ is adjacent to exactly 2 vertices of $N_1(u)$. Hence, a vertex $w$ of $N_1(u)$ is adjacent to $w_1$ and $w_2$ and has eccentric vertices in $N_1(u)$ only, and for $v \in N_1(u)$, $v$ is adjacent to one of $w_1$ or $w_2$ and is adjacent to $u$. Hence, degree of $v$ in $N_1(u)$ is $3-1-1=1$. $N_1(u)$ contains 3 elements. Therefore, $v$ is not adjacent to exactly one element $w$ of $N_1(u)$. Hence, $\{u, w\}$ is an eccentric dominating set of $G$. Therefore, $\gamma_{ed}(G) = 2$.

![Figure 2.3](image-url)
Remark: 2.1: Let $G$ be a $(p-3)$ regular graph with $p > 5$. Then, by Theorem 1.10, $\gamma_{ed}(G) \leq p/2$, since in this case $G = \overline{C_n}$.

Theorem: 2.4: Let $G$ be a connected $p-4$ regular graph. Then $\gamma_{ed}(G) \leq p/2$ for $p \geq 6$.

Proof: $G$ is a $p-4$ regular graph. Therefore $p$ is even.

When $p = 6$, $G = C_6$ and $\gamma_{ed}(G) = 3 = p/2$.

Let $u \in V(G)$. Since $G$ is $(p-4)$ regular, $N_1(u)$ contains $(p-4)$ vertices and $N_2(u)$ contains exactly three vertices. (If $N_2(u)$ contains two vertices then $N_3(u)$ contains one vertex, whose degree is one or two, a contradiction). Hence, $G$ is two self-centered, when $p \geq 8$.

Now, we claim that $<N_1(u)>$ is not complete.

Suppose, $<N_1(u)>$ is complete, $<N_1(u)> = K_{p-4}$ and if $v \in N_1(u)$ then $v$ is adjacent to $u$. This implies that, $\deg v \geq p-5+1 = p-4$. But $\deg v = p-4$ implies that $v$ has no adjacent vertices in $N_2(u)$. This is true for all $v \in N_1(u)$, which is a contradiction, since $G$ is connected. Hence, $<N_1(u)>$ is not complete.

Let $N_2(u) = \{w_1, w_2, w_3\}$. Since $\deg w_1 = p-4$, it is adjacent to all vertices of $N_1(u)$; or is adjacent to $w_2$ or $w_3$ and any $p-5$ vertices of $N_1(u)$; or is adjacent to both $w_2$ and $w_3$ and any $p-6$ vertices of $N_1(u)$.

Case (i): $<N_2(u)>$ is independent.

$w_1$, $w_2$ and $w_3$ are pair wise disjoint. Therefore, $w_1$ is adjacent to all vertices of $N_1(u)$ and is eccentric to $u$, $w_2$ and $w_3$. Similarly, $w_2(w_3)$ is adjacent to all of $N_1(u)$ and is eccentric to $u$, $w_1$ and $w_3(w_2)$. Hence, elements in $N_1(u)$ has eccentric points in $N_1(u)$ only and for $v \in N_1(u)$, $v$ is adjacent to $w_1$, $w_2$ and $w_3$ and $v$ is adjacent to $u$. Hence, degree of $v$ in $N_1(u) = p-4-3-1 = p-8$. But $N_1(u)$ contains $p-4$ elements. Hence, $v$ is not adjacent to exactly three elements of $N_1(u)$. Therefore, at most $(p-4)/2$ vertices from $N_1(u)$ is needed to dominate $G$ eccentrically. $\{u\} \cup S$, where $S \subseteq N_1(u)$ containing $(p-4)/2$ such vertices is an eccentric dominating set. Hence, $\gamma_{ed}(G) \leq 1+(p-4)/2 \leq p/2$.

Case (ii): $<N_2(u)> = K_2 \cup K_1$, $w_1$ and $w_2$ are adjacent.

In this case, $w_1$ and $w_2$ are adjacent to exactly $p-5$ vertices of $N_1(u)$ and is eccentric to $u$ and $w_3$ is adjacent to all vertices of $N_1(u)$ and is eccentric to $u$, $w_1$ and $w_2$. Hence, elements in $N_1(u)$ has eccentric vertices in $N_1(u)$. Let $v \in N_1(u)$. Consider the following sub cases:

Sub Case (i): $v$ is adjacent to $w_1$ and $w_2$ and $v$ is adjacent to $u$. 

Sub Case (ii):...
In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-1-3 = p-8 \). But \( N_1(u) \) contains \( p-4 \) elements. Therefore, \( v \) is not adjacent to exactly three elements in \( N_1(u) \).

**Sub Case (ii):** \( v \) is adjacent to \( w_1 \) or \( w_2 \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-1-1-1 = p-7 \). But \( N_1(u) \) contains \( p-4 \) elements. Therefore, \( v \) is not adjacent to exactly two elements in \( N_1(u) \). Therefore, at most \( (p-4)/2 \) vertices from \( N_1(u) \) is needed to dominate \( G \) eccentrically. \( \{u\} \cup S \), where \( S \subseteq N_1(u) \) containing \( (p-4)/2 \) such vertices is an eccentric dominating set. Hence, \( \gamma_{ed}(G) \leq 1+(p-4)/2 \leq p/2 \).

**Case (iii):** \( \langle N_2(u) \rangle = K_3 \). \( w_1, w_2 \) and \( w_3 \) are adjacent to each other.

\( \langle N_2(u) \rangle \) is complete. Therefore, \( w_1, w_2 \) and \( w_3 \) are adjacent to exactly \( p-6 \) vertices of \( N_1(u) \) and is eccentric to \( u \). Let \( v \in N_1(u) \), consider the following sub cases:

**Sub Case (i):** \( v \) is adjacent to \( w_1 \), \( w_2 \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-2-1 = p-7 \). Therefore, \( v \) is not adjacent to exactly two vertices in \( N_1(u) \).

**Sub Case (ii):** \( v \) is adjacent to \( w_1 \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-1-1 = p-6 \). Therefore, \( v \) is not adjacent to exactly one vertex in \( N_1(u) \).

**Sub Case (iii):** \( v \) is adjacent to all the three vertices of \( N_2(u) \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-3-1 = p-8 \). Therefore, \( v \) is not adjacent to exactly three vertices in \( N_1(u) \).

Therefore, at most \( (p-4)/2 \) vertices from \( N_1(u) \) is needed to dominate \( G \) eccentrically. \( \{u\} \cup S \cup \{w_1\} \), where \( S \subseteq N_1(u) \) containing \( (p-4)/2 \) such vertices is an eccentric dominating set. Hence, \( \gamma_{ed}(G) \leq 1+(p-4)/2 = p/2 \).

**Case (iv):** \( \langle N_2(u) \rangle = K_{1,2} \). \( w_2 \) is adjacent to \( w_1 \) and \( w_3 \).

In this case, \( w_1 \) and \( w_3 \) are adjacent to exactly \( p-5 \) vertices of \( N_1(u) \) and is eccentric to \( u \), \( w_2 \) is adjacent to exactly \( p-6 \) vertices of \( N_1(u) \) and is eccentric to \( u \). Let \( v \in N_1(u) \). Consider the following sub cases:

**Sub Case (i):** \( v \) is adjacent to \( w_1 \) or \( w_2 \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-1-1-1 = p-7 \). But \( N_1(u) \) contains \( p-4 \) vertices. Hence, \( v \) is not adjacent to exactly two elements of \( N_1(u) \).

**Sub Case (ii):** \( v \) is adjacent to \( w_1 \) and \( w_2 \) and \( v \) is adjacent to \( u \).

In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-2-1 = p-7 \). Therefore, \( v \) is not adjacent to exactly two vertices in \( N_1(u) \).

**Sub Case (iii):** \( v \) is adjacent to all the three vertices of \( N_1(u) \).
In this case, degree of \( v \) in \( N_1(u) \) is \( p-4-3-1 = p-8 \). Therefore, \( v \) is not adjacent to exactly three vertices in \( N_1(u) \).

Hence, at most \( (p-4)/2 \) vertices from \( N_1(u) \) is needed to dominate \( G \) eccentrically. 
\[ \{u\} \cup \mathcal{S} \cup \{w_2\}, \text{ where } \mathcal{S} \subseteq N_1(u) \text{ containing } (p-4)/2 \text{ such vertices is an eccentric dominating set. Hence, } \gamma_{ed}(G) \leq 1+1+(p-4)/2 = p/2 \]

So, in all the cases, \( \gamma_{ed}(G) \leq p/2 \). Hence the theorem follows.

**Remark:** Theorem 2.4 can also be proved using Lemma: 2.5

**Corollary: 2.4:** If \( G \) is a connected 4-regular graph on 8 vertices, then \( \gamma_{ed}(G) \leq 4 \).

In the following theorems, we have characterized graphs for which \( \gamma_{ed}(G) = p-1, p-2 \), where \( p \) is the number of vertices of \( G \).

**Theorem:** 2.5: Let \( G \) be a connected graph. Then \( \gamma_{ed}(G) = p-1 \) if and only if \( G \cong K_2 \) or \( K_{1,2} \).

**Proof:** If \( G = K_2 \), then \( \gamma_{ed}(G) = 1 \). Hence, \( \gamma_{ed}(G) = p-1 \). If \( G = K_{1,2} \), then \( \gamma_{ed}(G) = 2 \). Hence, \( \gamma_{ed}(G) = p-1 \).

Conversely, assume that, \( \gamma_{ed}(G) = p-1 \). Then there exists an eccentric dominating set \( D \) containing \( p-1 \) vertices. By Theorem 1.9, \( \gamma_{ed}(G) \leq (2p)/3 \). Therefore, we get, \( p-1 \leq (2p)/3 \). That is \( 3p-3 \leq 2p \). Therefore, \( p \leq 3 \) and it follows that, \( G \cong K_2 \) or \( K_{1,2} \).

**Theorem:** 2.6: Let \( G \) be a connected graph. Then \( \gamma_{ed}(G) = p-2 \) if and only if \( G \) is any one of the following graphs: \( K_3, C_4, P_5, C_5-e, K_{1,3}, K_1+K_1+K_2, K_1+K_1+K_1+2K_1, \) Bull graph, Fan graph\( (F_4), \overline{K_2}+K_1+K_1+\overline{K_2} \).

**Proof:** For all graphs in the theorem, \( \gamma_{ed}(G) = p-2 \).

Conversely, let \( G \) be a connected graph with \( \gamma_{ed}(G) = p-2 \). By Theorem 1.9, \( \gamma_{ed}(G) \leq (2p)/3 \).

Thus, we get \( p-2 \leq (2p)/3 \). That is \( 3p-6 \leq 2p \). Therefore, \( p \leq 6 \), and it follows that, \( G \) is one of the graph in the stated theorem.

Next, necessary condition for \( \gamma_{ed}(G) = p-k \), where \( k = 1, 2, 3, \ldots \) is given.

**Theorem:** 2.7: Let \( G \) be a connected graph. If \( \gamma_{ed}(G) = p-k \), where \( k = 1, 2, 3, \ldots \), then \( p \leq 3k \).

**Proof:** By Theorem 1.9, \( \gamma_{ed}(G) \leq (2p)/3 \). Thus, we get \( p-k \leq (2p)/3 \). That is \( 3p-3k \leq 2p \). This implies that \( p \leq 3k \).
Theorem: 2.8: Let $G$ be a graph with $p \leq 6$ vertices. Then $\gamma_{ed}(G) = p-3$ if and only if $G$ is any one of the graphs given in Figure 2.4 a and Figure 2.4 b.

Proof: Let $G$ be a connected graph with $\gamma_{ed}(G) = p-3$. By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Thus, we get, $p-3 \leq (2p)/3$. That is $p \leq 9$. When $p \leq 6$, the graphs given in Figure 2.4 a and Figure 2.4 b are the only graphs with $\gamma_{ed}(G) = p-3 = 3$.

In the following theorem, the upper bound of $\gamma_{ed}(G)$ is obtained in terms of order and size of $G$. 

Figure 2.4 a
**Theorem 2.9:** For any connected graph $G$ with $p \geq 2$, $\gamma_{ed}(G) \leq 2q-p+1$. Further, the equality holds if and only if $G \cong K_2$ or $K_{1,2}$.

**Proof:** For any graph $G$, $\gamma_{ed}(G) \leq p-1 = 2(p-1)-(p-1) \leq 2q-p+1$. 
Further, if \( \gamma_{ed}(G) = 2q - p + 1 \), then \( 2q - p + 1 \leq p - 1 \) and \( q \leq p - 1 \). Hence, \( q = p - 1 \).

Therefore, \( G \) is a tree. By Theorem 2.4, \( \gamma_{ed}(G) \cong K_2, K_{1,2} \). Hence, the result is proved.

**Theorem 2.10:** For any connected \((p, q)\) graph \( G \), \( \gamma_{ed}(G) + \Delta(G) \leq 2p - 2 \) \((p \geq 2)\). Also, the equality holds if and only if \( G \cong K_2, K_{1,2} \).

**Proof:** For any graph with \( p \) vertices, \( \Delta(G) \leq p - 1 \) and \( \gamma_{ed}(G) \leq p - 1 \). Hence, \( \gamma_{ed}(G) + \Delta(G) \leq 2p - 2 \). When \( G \cong K_2 \) or \( K_{1,3} \), \( \gamma_{ed}(G) + \Delta(G) = 2p - 2 \).

Conversely, assume that \( \gamma_{ed}(G) + \Delta(G) = 2p - 2 \). The only possibility is \( \gamma_{ed}(G) = p - 1 \) and \( \Delta(G) = p - 1 \). Hence, by Theorem 2.4, \( G \cong K_2 \) or \( K_{1,2} \).

**Theorem 2.11:** For any connected \((p, q)\) graph \( G \), \( \gamma_{ed}(G) + \Delta(G) = 2p - 3 \) if and only if \( G \) is any one of the following: \( K_3, K_4 - e, K_{1,3}, K_1 + K_1 + K_2 \), and Fan graph \( F_v \).

**Proof:** For the graphs given in the theorem, \( \gamma_{ed}(G) + \Delta(G) = 2p - 3 \).

Conversely, assume that \( \gamma_{ed}(G) + \Delta(G) = 2p - 3 \). The possible cases are

(i) \( \gamma_{ed}(G) = p - 1 \) and \( \Delta(G) = p - 2 \) and
(ii) \( \gamma_{ed}(G) = p - 2 \) and \( \Delta(G) = p - 1 \).

**Case (i):** By Theorem 2.4, \( \gamma_{ed}(G) = p - 1 \) if and only if \( G \cong K_2 \) or \( K_{1,2} \).

In this case, \( \Delta(G) = p - 1 \). Therefore, this case is not possible.

**Case (ii):** By Theorem 2.5, \( \gamma_{ed}(G) = p - 2 \) if and only if \( G \) is any one of the following graphs:

- \( K_3, C_4, P_5, C_5, K_4 - e, K_{1,3}, K_1 + K_1 + K_2, K_1 + K_1 + K_1 + 2K_1 \), Bull graph, Fan graph \( F_v \), \( \overline{K_2 + K_1 + K_2} \).

If \( G \) is any one of the graphs \( C_4, P_5, C_5, K_1 + K_1 + K_1 + 2K_1 \), Bull graph, \( \overline{K_2 + K_1 + K_1 + K_2} \), then \( \Delta(G) \neq p - 1 \). These cases are not possible. Therefore, \( G \) is one of the graphs in the stated theorem.

In the following, Nordhaus-Gaddum type results for eccentric domination number are established.

**Theorem 2.12:** Let \( G \) be a \((p, q)\) \((p \geq 4)\) graph such that both \( G \) and its complement \( \overline{G} \) are connected. Then (i) \( 4 \leq \gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 2(p - 2) \). (ii) \( 4 \leq \gamma_{ed}(G) \cdot \gamma_{ed}(\overline{G}) \leq 2(p - 2)^2 \).

**Proof:** (i) By Theorem 2.4, \( \gamma_{ed}(G) = p - 1 \) if and only if \( G \cong K_2 \) or \( K_{1,2} \). But in this case, \( \overline{G} \) is disconnected. Therefore, \( \gamma_{ed}(G) \leq p - 2 \). Hence, \( \gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 2(p - 2) \).
For the lower bound, $\gamma_{ed}(G) = 1$ if and only if $G$ is a complete graph. In this case, $\overline{G}$ is disconnected. Therefore, $\gamma_{ed}(G) \geq 2$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \geq 4$.

(ii) Proof follows similarly.

Lower bound is attained, if $G$ is a path on 4 vertices and upper bound is attained, if $G$ is a cycle on 5 vertices.

**Theorem: 2.13:** Let $G$ be a $(p, q)$, $p \geq 4$ graph such that both $G$ and its complement $\overline{G}$ are connected. Then (i) $4 \leq \gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 4p/3$. (ii) $4 \leq \gamma_{ed}(G) \cdot \gamma_{ed}(\overline{G}) \leq 4p^2/9$.

**Proof:** (i) By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 4p/3$. For the lower bound, $\gamma_{ed}(G) = 1$ if and only if $G$ is a complete graph. In this case, $\overline{G}$ is disconnected. Therefore, $\gamma_{ed}(G) \geq 2$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \geq 4$.

(ii) Proof is similar as in case (i).

We have characterized graphs for which $\gamma(G) = \gamma_{ed}(G) = p/2$ in the following theorem:

**Theorem: 2.14:** For a connected graph $G$ with even number of vertices $p$ and $\delta(G) \geq 2$, $\gamma(G) = \gamma_{ed}(G) = p/2$ if and only if $G$ is $C_4$ or $H \circ K_1$ for some connected graph $H$.

**Proof:** When $G = C_p$, $\gamma_{ed}(G) = 2 = p/2$. Let $G = H \circ K_1$, where $H$ is a connected graph on $p/2$ vertices. $V(H)$ is a $\gamma$-set for $G$ and the set of all pendant vertices in $G$ is a minimum eccentric dominating set. Hence, $\gamma_{ed}(G) = p/2$.

Conversely, assume that $\gamma(G) = \gamma_{ed}(G) = p/2$. Since $G$ is a graph with $\delta(G) \geq 2$, $p$ is even, by Theorem 1.1, we get $G$ is $C_4$ or $H \circ K_1$ for some connected graph $H$.

**Theorem: 2.15:** (i) $G$ is a connected graph with $\delta(G) \geq 2$ and $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$ if and only if $G$ is any one of the graphs given in Figure 2.5.

(ii) $G$ is a connected graph with $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor + 1$ if and only if $G$ is $C_5$.

**Proof:** (i) Let $G$ be a connected graph with $\delta(G) \geq 2$ and $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$. By Theorem 1.2, $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ implies $G$ is any one of the graphs in Figure 1.1. Therefore, the graphs given in Figure 2.5 are the only graphs with $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$. 


(ii) Suppose $G$ is $C_5$. It is clear that $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lceil \frac{p}{2} \right\rceil + 1$.

Conversely, assume that $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lceil \frac{p}{2} \right\rceil + 1$. By Theorem 1.2, $C_5$ is the only graph with $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lceil \frac{p}{2} \right\rceil + 1$.

References:


